PHYSICS

XI

UNIT 4

ROTATIONAL AND CIRCUAR MOTION

PROF:IMRAN HASHMI

ROTATIONAL MOTION

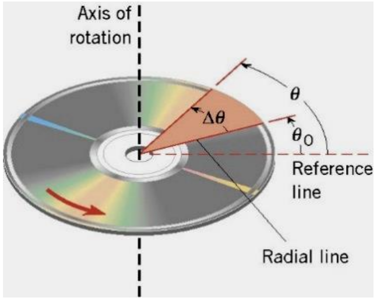
Rotational motion is defined as the movement of an item in a fixed orbit around a circular path. It can be seen in practically everything around us. Every machine, celestial bodies, and the majority of games actually do rotating motion.

ANGULAR DISPLACEMENT

Angle subtended by a body moving in a circle at the center of its circular path is called its angular displacement

UNITS:

Angular displacement can be measured in the following unit



1. Degree (deg)　　2. Revolutions (rev)
3. Radians (rad)

The last unit radian is in common use for the measurement of angular displacement.

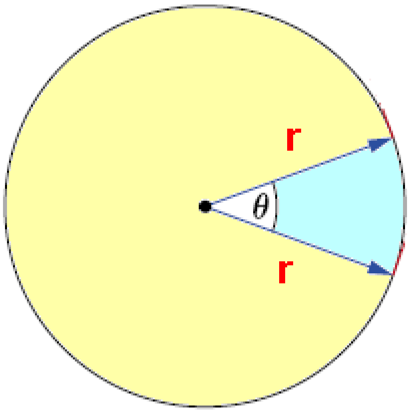
DIMENSIONS

The measure of an angle in degrees, revolution or radian, does not have physical dimensions therefore it is dimensionless quantity.

RADIAN

Radian is the angle subtended at the center of the circle by an arc equal in length to its radius

RELATION B/W RADIAN AND DEGREES



Consider the fig. In which

r = radius of the circle
S = arc length
θ = angle subtended by arc 'S'

the relation between and is given by

$S = r\theta$ ———— (1)

when a body rotates in a complete circle, it describes an arc of one circumference, i.e.

$S = 2\pi r$ ———— (ii)

Comparing equation (i) and (ii)

$2\pi r = r\theta$

$2\pi = \theta$

but in a complete circle there are 360°, therefore $\theta = 360^o$

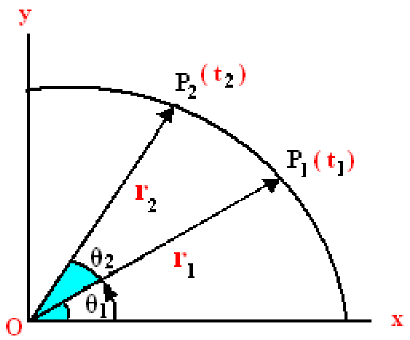
1 revolution= 2 π rad = 360°

Now, 2π rad = 360° **1 rad = $\frac{360}{2\pi}$** **1 rad = 57.3 °**	**2π rad = 1rev** **1 rad=$\frac{1}{2\pi}$rev** **1 rad = 573°** **1 rad = 0.159 rev**

ANGULAR VELOCITY:

The rate of change of angular displacement is called angular velocity or angular frequency .It is a vector quantity represented by

FORMULA :



Consider a body moving in a circle of radius r counterclockwise as shown

Let θ_1 = angular position at time t_1

θ_2 = angular position at time t_2

angular displacement during interval$(t_2 - t_1)$

$$\Delta\theta = \theta_2 - \theta_1$$

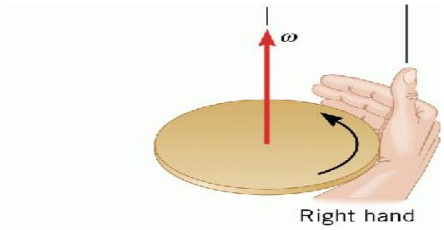
$\Delta t = t_2 - t_1$ time interval

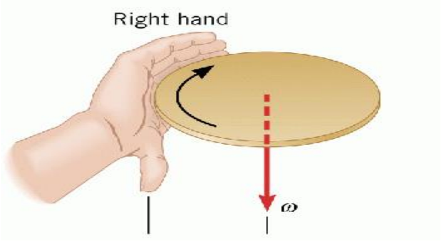
Now the average angular velocity is defined as the ratio of angular displacement to time, interval Δt.

$$\omega_{av} = \frac{\Delta \theta}{\Delta t}$$

The instantaneous angular velocity is the limit of the ratio as approaches zero

$$\omega = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\theta}{\Delta t}$$





DIRECTION:

The direction of angular velocity is given by right hand rule, which is defined as " curl fingers of right hand around the axis of rotation, in the direction of rotation then right hand thumb will point in the direction of angular velocity ω

UNITS:

Its units are

(i) rad / sec

(ii) deg / sec

(iii) rev/sec

DIMENSIONS:

$$\omega_{av} = \frac{\Delta\theta}{\Delta t} = \frac{1}{T} = T^{-1}$$

It has dimensions of inverse time [T^{-1}]

ANGULAR ACCELERATION

The rate of change of angular velocity is called angular acceleration. It is a vector quantity represented by $\vec{\alpha}$

FORMULA:

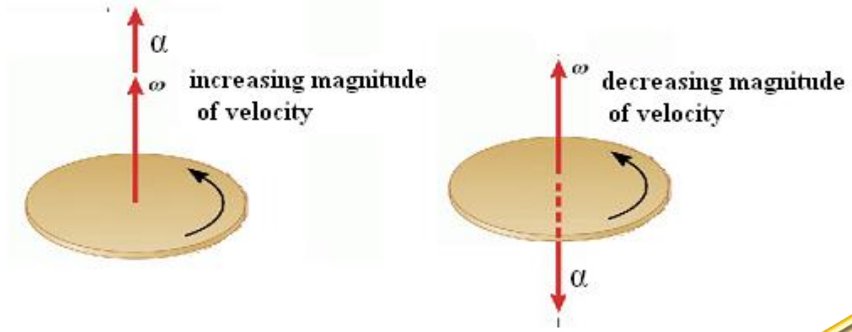
$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The instantaneous angular acceleration is the limit of this ratio as Δt approaches to zero:

$$\alpha = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\omega}{\Delta t}$$

DIRECTION:

1. The angular acceleration is parallel to for increasing magnitude of angular velocity
2. The angular acceleration is opposite to for decreasing magnitude of angular velocity



UNITS:

Its units are in
(i) rad / s^2
(ii) rev / s^2
(iii) deg / s^2

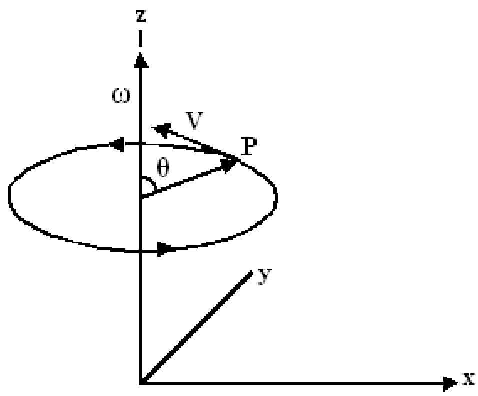
DIMENSIONS:

$$\because \alpha_{av} = \frac{\Delta\omega}{\Delta t} = \frac{T^{-1}}{t} = T^{-2}$$

Angular acceleration has the dimensions of an inverse time squared (T^{-2}).

RELATION BETWEEN LINEAR VELOCITY $\vec{v}$ AND ANGULAR VELOCITY $\vec{\omega}$:

Consider a particle 'P' in a body rotating in a circle of radius 'r' as shown.
Suppose the particle 'P' moves through a distance along the arc when the body rotates through a distance along the arc when the body rotates through an angle, such that



$$\Delta S = r\,\Delta\theta$$

Dividing both side by the time interval in which the rotation occurred, we get

$$\frac{\Delta S}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

Applying $\underset{\Delta t \to 0}{Limit}$ on both sides, above eqn becomes

$$\underset{\Delta t \to 0}{\text{Limit}}\left(\frac{\Delta S}{\Delta t}\right) = \underset{\Delta t \to 0}{\text{Limit}} = \left(\frac{\Delta\theta}{\Delta t}\right) \ldots\ldots\ldots\ldots(i)$$

Where $\underset{\Delta t \to 0}{\text{Limit}} = \frac{\Delta S}{\Delta t} = V$ = instantaneous linear velocity

$\underset{\Delta t \to 0}{\text{Limit}} = \frac{\Delta \theta}{\Delta t} = \omega$ = instantaneous angular velocity

substituting these values in equation (1) we get

$$V = r\,\omega$$

where **r** is perpendicular distance from 'P' to the axis of rotation

TANGENTIAL VELOCITY

The tangential velocity of a particle moving in a circle is the product of the perpendicular distance from the axis of rotation to the particle and the angular velocity."

$V = r\,\omega$ Scalar form

$\vec{V} = \vec{\omega} \times \vec{r}$ Vector form

RELATION BETWEEN LINEAR AND ANGULAR ACCELERATION

Suppose an object rotating in a circle, changes angular velocity by in a time. then the change in its linear (tangential) velocity is given by

$$\Delta V = r\,\Delta\omega$$

Dividing both sides by Δt ,we get

$$\frac{\Delta v}{\Delta t} = r\,\frac{\Delta\omega}{\Delta t}$$

Applying $\lim_{\Delta t \to 0}$ on both sides , the above equation becomes

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} \quad ....(i)$$

But $\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = a$ **instantaneous linear acceleration**

$\lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \alpha$ **instantaneous angular acceleration**

Substituting these values in equation (i) we get

$$\mathbf{a} = \alpha\,\mathbf{r}$$

TENGENTIAL ACCELERATION:

Tangential acceleration a_t corresponding to tangential velocity V_t is the product of perpendicular distance from the axis of rotation to the particle and angular acceleration.

$$\mathbf{a_t = r\,\alpha}$$

THE PERIOD

DEFINITION:

The time required by a body for a complete revolution or cycle of the motion is known as time periods.

FORMULA:

$$T = \frac{2\pi}{\omega}$$

UNIT

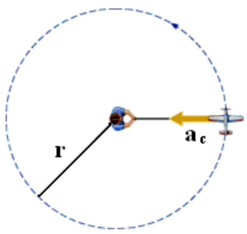
It is measured in seconds

DIMENSIONS

[T] It has a dimension of time

CENTRIPETAL ACCELERATION

DEFINITION:



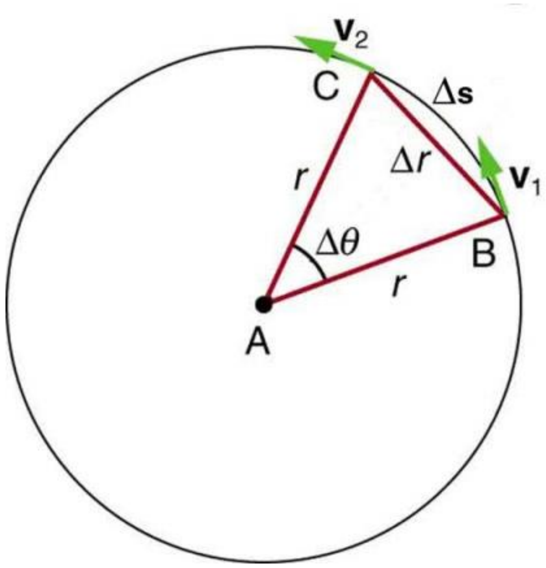
uniform circular motion

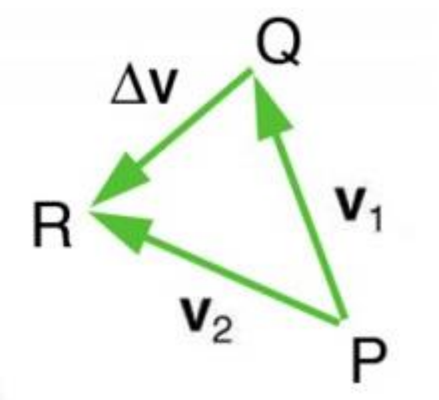
The acceleration produced due to the changing direction of velocity of a body moving in a circular path with constant speed is called centripetal acceleration.

It is a vector quantity always directed toward the center of circular path.

FORMULA

Consider a particle moving in a circle of radius 'r' with constant speed 'v'. The velocity at point 1 is a vector $\vec{v}_1$ tangent to the curve at point 1, and velocity at point 2 is $\vec{v}_2$ as shown in the figure.





for uniform circular motion

$$\vec{v_1} = \vec{v_2} = V$$

Consider two similar triangles ABC and PQR, Hence using the property of similar triangles

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}$$

$$\Delta v = v \; \frac{\Delta s}{r}$$

Let Δt be the time interval in which particle moves from point 1 to point 2 . Directing both sides by Δt we have

$$\frac{\Delta v}{\Delta t} = \left(\frac{v}{r}\right) \frac{\Delta s}{\Delta t}$$

When Δt is very small,. then applying limit $\underset{\Delta t \to 0}{Limit}$ on both sides, we find

$$\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta v}{\Delta t} = \left(\frac{v}{r}\right) \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta s}{\Delta t} \quad \text{................(i)}$$

In this equation ,

$$\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta v}{\Delta t} = a_c$$ **=centripetal acceleration.**

$$\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta S}{\Delta t} = V$$

then equation (i) becomes

$$a_c = \left(\frac{v}{r}\right) .v$$

$$a_c = \frac{v^2}{r}$$

CENTRIPETAL ACCELERATION IN TERM OF ANGULAR VELOCITY

we know that

$$v = \omega r \quad ....(i)$$

put $v = \omega r$ **in above equation**

$$a_c = \frac{(\omega\ r)^2}{r}$$

$$a_c = \frac{\omega^2\ r^2}{r}$$

$$a_c = \omega^2\ r$$

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CENTRIPETAL ACCELERATION IN TERM OF PERIOD

$$a_c = r\left(\frac{2\pi}{T}\right)^2 \qquad \left(\omega = \frac{2\pi}{T}\right)$$

$$a_c = \frac{4\pi^2 r}{T^2}$$

UNITS:

The units of centripetal acceleration are the same as those of an acceleration i.e. m/s^2

DIMENSIONS:

LT^{-2} are the dimension of centripetal acceleration

CENTRIPETAL FORCE

DEFINITION:

The Force which is required to produce centripetal acceleration in a body moving in circle is known as centripetal force. It is a vector quantity represented by $\vec{F}$

FORMULA:

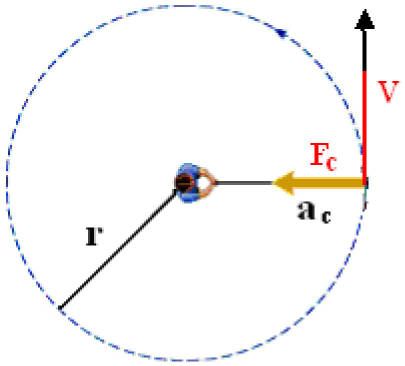
Consider a body of mass 'm' moving in a circle of radius 'r' with uniform speed . The direction of velocity is changing continuously towards the center of the circle . The force responsible for the change in direction of the velocity in the circle .

By Newton's second Law.

$$F_c = m\ a_c$$

CENTRIPETAL FORCE IN TERM OF LINEAR VELOCITY

$$F_c = m\ \frac{v^2}{r} \qquad \left[\ a_c = \frac{v^2}{r}\ \right]$$



uniform circular motion

CENTRIPETAL FORCE IN TERM OF ANGULAR VELOCITY

$$F_c = m\ \frac{(r\ \omega)^2}{r} \qquad [\ v = r\,\omega\ ]$$

$$F_c = m\ \frac{r^2\ \omega^2}{r}$$

$$F_c = m\ r\ \omega^2$$

UNIT:

In SI system of units it is measured in 'newton' (N).

DIMENSIONS

MLT^{-2} are the dimension of force

NON-UNIFROM CIRCULAR MOTION

If the speed of a body revolving in a circle is changing, there will be tangential acceleration a_{tan}, as well as the centripetal accelerations a_c. The tangential acceleration arises from the change in the magnitude of the velocity.

$$a_{tan} = \frac{\Delta v}{\Delta t}$$

whereas the centripetal acceleration arises from the change the direction of velocity.

$$a_c = \frac{v^2}{r}$$

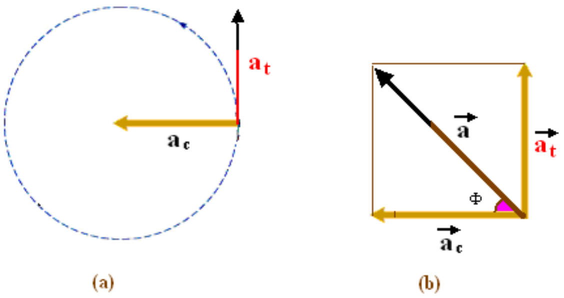
These two component of acceleration are perpendicular to each other, then total acceleration $\vec{a}$, by using vector diagram in figure (b), is given by

$$\vec{a} = \vec{a}_c + \vec{a}_t$$

The magnitude of $\vec{a}$ is determined by Pythagoras theorem

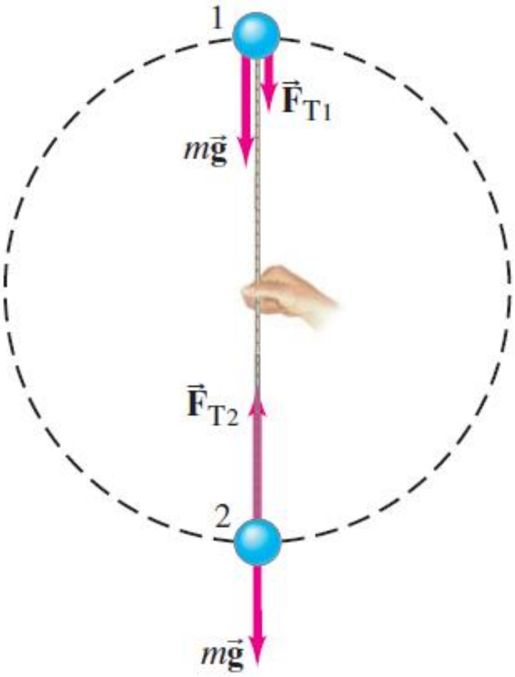
$$a = \sqrt{a_c^2 + a_t^2}$$

The direction is given by $\Phi = \text{Tan}^{-1}\left(\frac{a_t}{a_c}\right)$



VERTICAL CIRCULAR MOTION

The ball moves in a vertical circle and is *not* undergoing uniform circular motion. The radius is assumed constant, but the speed v changes because of gravity. we consider the top and bottom points.



At the top (point 1)

At the top (point 1), two forces act on the ball: the force of gravity F_w = mg, and the tension F_T force the cord exerts at point 1. Both act downward. Newton's second law, for the vertical direction, choosing downward as positive

$$F_T + F_W = F_c$$

$$F_T + mg = \frac{m v^2}{r}$$

$$F_T = \frac{m v^2}{r} - mg$$

At the bottom (point 2)

When the ball is at the bottom of the circle (point 2), the cord exerts its tension force F_T upward, whereas the force of gravity F_w = mg, still acts downward. Choosing *upward* as positive, Newton's second law gives

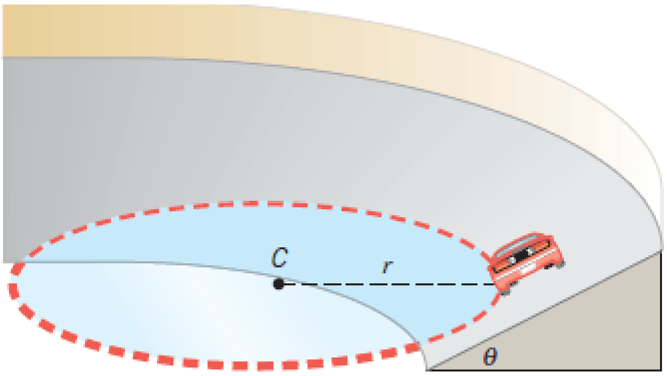
$$F_T - F_W = F_c$$

$$F_T - mg = \frac{m v^2}{r}$$

$$F_T = \frac{m v^2}{r} + mg$$

FORCES ACTING ON BANKED CURVE

A banked curve is a curve that has its surface at angle with respect to the ground on which the curve is positioned as shown in figure. The reason for banking curves is to decreases the moving object depends on the force of friction.



On a banked curve however, the normal force acting on the object such as a car, will act at an angle with the horizontal and that will create a component normal force that acts along the x axis. This component normal force will now be responsible for creating the centripetal acceleration required to move the car along the curve.

BANKING DEPENDENCE ON ANGLE AND SPEED OF VEHICLE.

Consider a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius r, then the net external farce equals the necessary centripetal force. The only two external forces acting on the car are its weight and the normal force of the road N. The normal force horizontal component must equal the centripetal force, that is.

$$N \sin\theta = \frac{m V^2}{r} \quad \dots\dots\dots\dots\dots (i)$$

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The vertical component of the normal force is Ncosθ, and the only other vertical force is the car's weight. These must be equal magnitude; thus,

N cosθ = mg …………(ii)

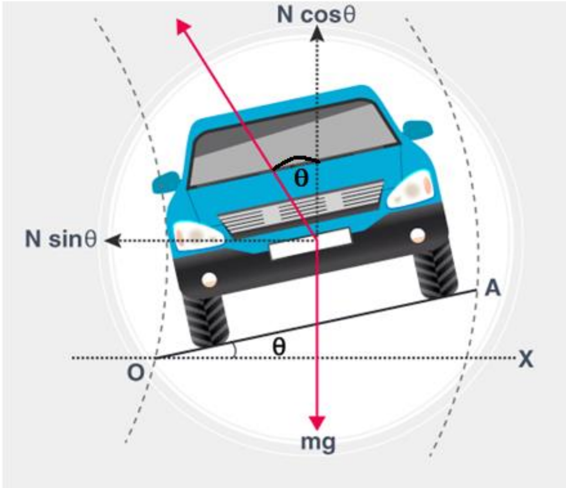
Dividing equation (i) by (ii)

$$\frac{N\sin\theta}{N\cos\theta} = \frac{\frac{mV^2}{r}}{mg}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{mV^2}{rmg}$$

$$\tan\theta = \frac{V^2}{rg}$$

$$\theta = \tan^{-1}\left(\frac{V^2}{rg}\right)$$



This expression can be understood by considering how θ depends on v and r. A large θ is obtained for a large v and a small r. That is, roads must be steeply banked far high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve were frictionless.

ORBITAL VELOCITY

Orbital velocity is the speed required to achieved orbit around a heavenly body, such as a planet or a star. We saw that there are natural satellites moving around the planets. There will be gravitational force between the planet and satellites. Nowadays many artificial satellites are launched into the Earth's orbit. The first artificial satellite Sputnik was launched in 1956

ORBITAL VELOCITY OF SATELLITE

Consider a satellite with mass M_{satt} is moving around the earth in an orbit of radius r and the velocity of the satellite is a tangent to the path of the orbit, and the net centripetal force acting on the satellite is .

$$F_c = \frac{M_{sat}\ V^2}{R} \dots\dots\dots\dots..(i)$$

This net centripetal force is the resultant of the gravitational force which attracts the Satellite towards the central body.

$$F_g = \frac{G\,M_{sat}\,M_E}{R^2} \dots\dots\dots\dots..(i)$$

Comparing equation (i) and (ii)

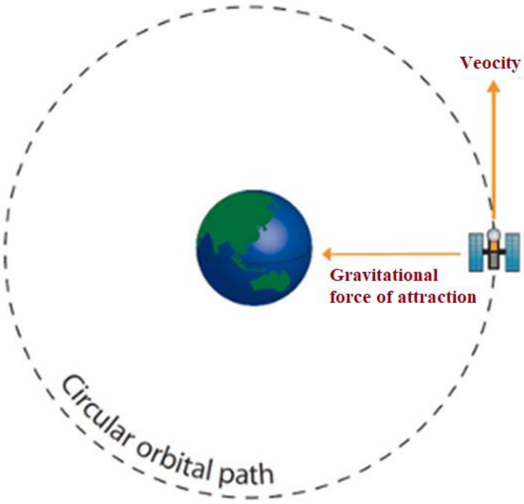
$$\frac{M_{sat}\ V^2}{r} = \frac{G\,M_{sat}\,M_E}{R^2}$$

$$\frac{V^2}{R} = \frac{G\ M_E}{R^2}$$

$$\frac{V^2}{R} = \frac{G\ M_E}{R\ R}$$

$$V^2 = \frac{G\ M_E}{R}$$

$$V = \sqrt{\frac{G\ M_E}{R}} \quad \dots.(iii)$$



TIME PERIOD AND ORBITAL RADIUS

A satellite is traveling in circular motion when in orbit, its orbital time periods T to travel the circumference of the orbit 2πr, the linear speed is

$$V = \frac{2\,\pi\, R}{T}$$

Substituting the expression of speed of satellite in equation (iii)

$$\frac{2\,\pi\, R}{T} = \sqrt{\frac{G\;M_E}{R}}$$

$$\frac{4\,\pi\, R^2}{T^2} = \frac{G\;M_E}{R}$$

$$\frac{4\,\pi\, R^3}{G\;M_E} = T^2$$

$$T^2 = \frac{4\,\pi\, R^3}{G\;M_E}$$

$$T = \sqrt{\frac{4\,\pi\, R^3}{G\;M_E}}$$

The equation shows that the orbital period T is related to the radius r of the orbit

MOMENT OF INERTIA

Moment of inertia is the property of the body by virtue of it resists angular acceleration, which is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation.

FORMULA

$$I = \sum m\; r^2$$

Where,

I= Moment of Inertia
m = Mass
r =distance to axis of rotation

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DIMENSION

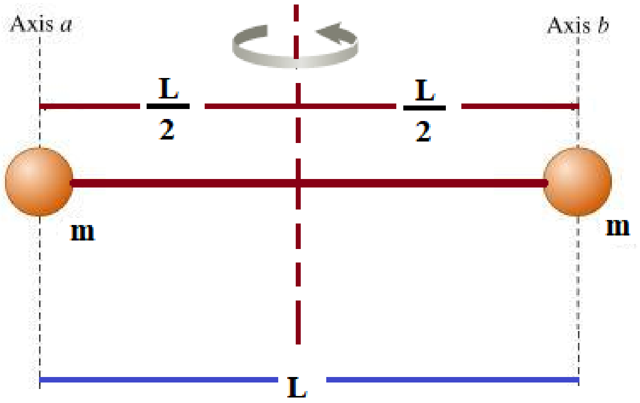
$M\,L^2$ are the dimension of moment of inertia

ROTATIONAL INERTIA OF A TWO PARTICLE SYSTEM

Consider a rigid body containing of two particles of mass m connected by arod of length L with negligible mass.

CASE -1

Two particles each at perpendicular distance 1/2 L from the axis of rotation. As shown in figure.



(a) **Two particles each at perpendicular distance 1/2 L from the axis of rotation.**
For two particles each at perpendicular distance 1/2 L from the axis of rotation, we have:

$$I = m\left(\frac{1}{2}L\right)^2 + m\left(\frac{1}{2}L\right)^2$$

$$I = m\,\frac{1}{4}L^2 + m\,\frac{1}{4}L^2$$

$$I = \frac{mL^2}{4} + \frac{mL^2}{4}$$

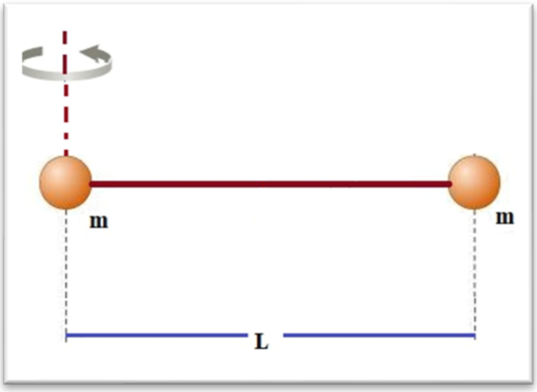
$$I = \frac{mL^2 + mL^2}{4}$$

$$I = \frac{2\,mL^2}{4}$$

$$I = \frac{mL^2}{2}$$

$$I = \frac{1}{2}mL^2$$

(b) **Rotational inertia I of the body about an axis through left end of rod and parallel to the first axis. The perpendicular distance is zero for the particle on left and L for the particle on right. We have:**

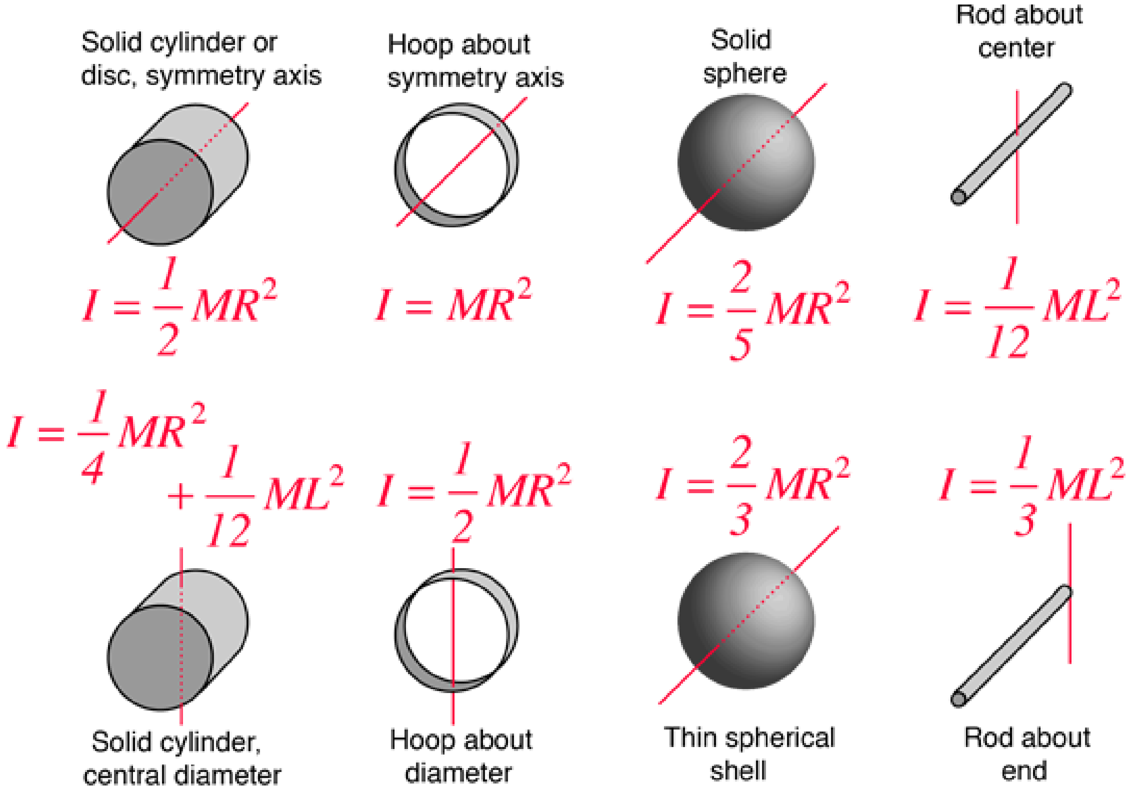


$$I = m(0)^2 + m(L)^2$$

$$I = 0 + m\,L^2$$

$$I = mL^2$$

<u>Moment of Inertia of Various bodies</u>



ANGULAR MOMENTUM

The angular momentum of a body about its axis is defined as the product of position vector ($\vec{r}$) and linear momentum. $\vec{p}$

Angular momentum is denoted by L

Consider a particle of mass m moving with a velocity $\vec{v}$ at a position $\vec{r}$ relative to the origin. The linear momentum of the particle is $\vec{p} = m\vec{v}$. The Angular momentum $\vec{L}$ of the particle relative to the origin is given by

$$\vec{L} = \vec{r} \times \vec{P}$$

If $\vec{r}$ and $\vec{p}$ x-y plane, as shown in the right figure then $\vec{L}$ is along the z axis

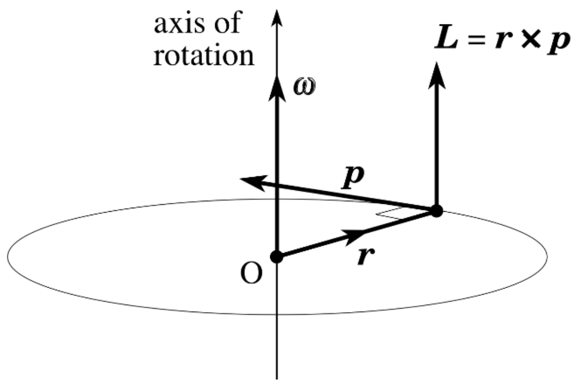
$$\vec{L} = \vec{r} \times \vec{P}$$

$$L = r\,p\,\mathrm{Sin}\,\theta$$

$$L = r\,(mv)\,\mathrm{Sin}\,\theta$$

$$\mathbf{L = r\ p\ sin\theta}$$

$$\mathbf{L = r\ (mv)\ sin\theta}$$



The angle between $\vec{r}$ and $\vec{v}$ is 90^0

$$L = r\,(mv)\,\mathrm{Sin}\,90^0$$

$$L = m v r$$

Consider a particle moving in a circle in the xy plane with the centre of the circle at the origin. The speed v of the particle and the magnitude of angular velocity ω are related by v = r ω. The angular momentum of the particle to the centre of the circle is

$$\mathbf{L = m\ v\ r}$$

$$\mathbf{L = m\ (r\ \omega)\ r}$$

$$\mathbf{L = m\ r^2\ \omega}$$

$$\vec{L} = \mathbf{m\ r^2\ \omega}$$

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If x, y, z are the component of position vector $\vec{r}$ and P_x, P_y, P_z are the component of momentum $\vec{p}$, then using the definition of vector product we write

$$\vec{L} = \vec{r} \times \vec{P}$$

$$\vec{L} = \begin{vmatrix} i & j & k \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix}$$

UNIT **The SI unit of momentum is Js (joule. second)**

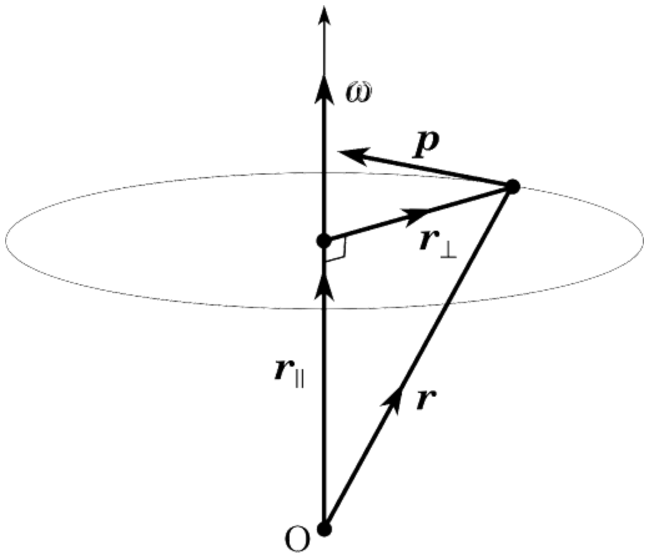
DIMENSION

The dimension of angular momentum are $L^2 M T^{-1}$

LAW OF CONSERVATION OF ANGULAR MOMENTUM

The total angular momentum of a system is constant on both magnitude and direction if the resultant external torque acting on the system is zero

PROOF



consider a particle of mass m moving with the velocity $\vec{V}$ along a circular path of radius r as shown in figure

The law of conservation of angular momentum then ensures that the angular momentum of the particle is constant. The angular momenti1m is constant in magnitude and constant indirection (the motion is confined to the plane of ro1ation).

It is worth emphasizing that the above discussion relies on the origin being at the center of the circle. In Figure 1 a different origin O has been chosen. on the axis of rotation but out of the plane of rotation. In this case angular momentum is

$$\vec{L} = \vec{r} \times \vec{P}$$

$$\vec{L} = (\vec{r}_{II} + \vec{r}_{\perp}) \times \vec{P}$$
$$\vec{L} = (\vec{r}_{II} \times \vec{P}) + (\vec{r}_{\perp} \times \vec{P})$$
$$\vec{L} = \overrightarrow{L_{\perp}} + \overrightarrow{L_{II}}$$
$$\overrightarrow{L_{\perp}} = (\vec{r}_{II} \times \vec{P}) \quad and \quad \overrightarrow{L_{II}} = (\vec{r}_{\perp} \times \vec{P})$$

ANGULAR MOMENTUM PERPENDICULAR TO AXIS OF ROTATION

The angular momentum has component $\overrightarrow{L_{\perp}} = (\vec{r}_{II} \times \vec{P})$ which is perpendicular to the axis of rotation and is not Conserved. This is not a problem because, relative to the origin of Figure 1, the particle experiences a torque $\overrightarrow{\tau_{\perp}} = (\vec{r}_{II} \times \vec{F})$ where F is the centripetal force acting along radius.

Moreover, because '$\vec{r}_{II}$ does not change constant we have

$$\overrightarrow{L_{\perp}} = (\vec{r}_{II} \times \vec{P})$$

The magnitude form of the angular momentum is

$$L_{\perp} = (r_{II}) \times (P)$$

Applying derivative on both the sides, we get

$$\frac{dL_{\perp}}{dt} = (r_{\,II}) \times \left(\frac{dP}{dt}\right)$$
$$\frac{dL_{\perp}}{dt} = (r_{\,II})\ (F)$$
$$\frac{dL_{\perp}}{dt} = \tau$$

The rate of change of angular momentum $L_{\perp}$ is supported by the existence of torque τ so the angular momentum is not conserved

ANGULAR MOMENTUM PARALLEL TO THE AXIS OF ROTATION

The component of angular momentum parallel to the axis of rotation remains Constant because there is no torque in that direction.

$$\overrightarrow{L_{II}} = (\vec{r}_{\perp} \times \vec{P})$$

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The magnitude form of the angular momentum is

$$L_{II} = (r_{\perp}) \times (P)$$

Applying derivative on both the sides, we get

$$\frac{d\,L_{II}}{dt} = (r_{\perp}) \times \left(\frac{dP}{dt}\right)$$

$$\frac{d\,L_{II}}{dt} = (r_{\perp})\ (F)$$

$$\frac{d\,L_{II}}{dt} = \tau$$

there is no torque in that direction

$$\frac{d\,L_{II}}{dt} = 0$$

Integrating both the side , we get

L = constant

Hence, the angular momentum of the particle is conserved if the net torque acting on it is zero

TORQUE

DEFINITION-1

The turning effect of an object due to an applied force is called "torque" or "moment of force"

We symbolized the moment of the force by a Greek letter ***tau*** (τ)

Torque (τ) depends upon

(i) Magnitude of the force applied (F)

(ii) The perpendicular distance of the line of action of the force from the axis of rotation (or fulcrum or pivot) also called "moment arm"(d)

FORMULA

$$\text{Moment of force} = \text{force} \times \left(\begin{matrix}\text{Perpendicular distance from the pivot} \\ \text{to the line of action of force}\end{matrix}\right)$$

$$\tau = F\,r$$

DIMENSION

The dimension of torque are ML^2T^{-2}

DEFINITION-1

Torque is defined as the cross product of moment arm and force

FORMULA

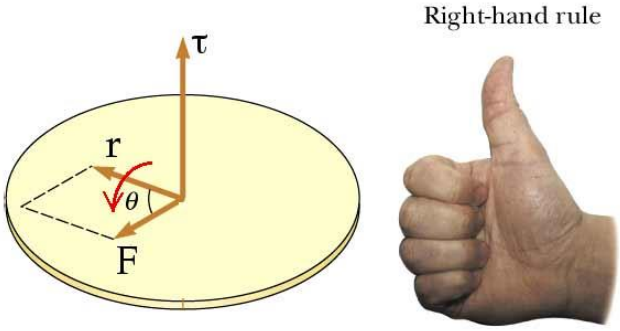
$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}|\ |\vec{F}|\ \sin\theta$$

$$\tau = r\ F\ \sin\theta$$

DIRECTION OF TORQUE

The nature of cross (or vector) product of two vectors, conveys a great deal about the direction of cross product i.e. torque in our case. It tells us that (i) torque vector is perpendicular to the plane formed by operand vectors i.e. "r" and "F" and (ii) torque vector is individually perpendicular to each of the operand vectors



CHARACTERISTICS

1. **Max. Torque: if the angle between r and F is 90°, then torque is maximum**

$$\tau_{max} = r F \sin 90^{o} = r F$$

2. **Min. Torque: If the angle between r and F is zero. (i.e. when a line of action of force and moment arm coincide), or if the direction of F is opposite to r (i.e. θ = 180°) then torque is minimum.**

$$\tau_{min} = r F \sin 0^{0} = zero \quad \text{also } \tau_{min} = r F \sin 180^{o} = zero$$

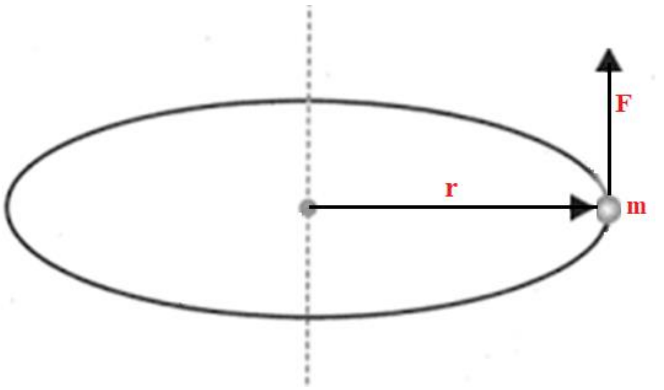
3. **If the direction of the force F is reversed, then the direction of torque is reversed, but its magnitude remains the same.**

4. **If the direction of r is reversed, then the direction of torque is reversed but its magnitude remains the same.**

5. **If the direction of both r and F are reversed, then neither the magnitude nor the direction of the torque will change.**

Derivation a relation torque, moment of inertia, and angular acceleration.

Consider a particle of mass m rotating in a circle of radius r at the end of a string whose mass is negligible as compared to the mass of the string. Assume that a single force F acts on mass as shown in figure



According to Newton's second law of motion

$$F = m a_t$$

The relation between tangential and linear acceleration is:

$$a_t = r \alpha$$

Equation (i) can be rewritten as

$$F = m r \alpha$$

Multiplying both sides by r

$$r \times F = m r \alpha \times r$$

$$r \times F = m r^2 \alpha \quad \ldots\ldots\ldots (ii)$$

we know that

$$\tau = r \times F \quad \text{and} \quad I = m r^2$$

Substituting the expression for torque and moment of inertia in equation (ii), we get

$$\tau = I \alpha$$

$$\tau \propto \alpha$$

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The above expression shows that the torque is direct relation to angular acceleration and of inertia is constant.

CHAPTER = 4 ROTATIONAL AND CIRCULAR MOTION

SHORTS QUESTIONS (BOOK- 11)

1: For an isolated rotating body, what is the relation between angular velocity and radius?

ANS: For an isolated rot acting body, the angular velocity is inversely proportional to the radius. This means that as the radius increases, the angular velocity decreases.
For an isolated rotating body with no external torques acting on it (i.e., angular momentum is conserved).
This relationship is described by the conservation of angular momentum.

$$I_1 \times \omega_1 = I_2 \times \omega_2$$

Where,

$$I = k \times r^2$$

Using the moment of inertia expression for both initial and final situations, we get:

$$k_1 \times r_1^2 \times \omega_1 = k_2 \times r_2^2 \times \omega_2$$

$$\frac{r_2^2}{r_1^2} = \frac{k_1}{k_2} \times \frac{\omega_1}{\omega_2}$$

2: When the moment of inertia of a rotating body is halved, then what will be the effect on angular velocity?

ANS: When the moment of inertia (I) of a rotating body is halved, the angular velocity (w) of the body will double, assuming no external torques act on the body. This is based on the principle of conservation of angular momentum.
The conservation of angular momentum can be expressed as:

$$I_1 \times \omega_1 = I_2 \times \omega_2$$

If we halve the moment of inertia ($I_2 = \frac{1}{2} I_1$), we can solve for ω_2.

$$I_1 \times \omega_1 = \frac{1}{2} I_1 \times \omega_2$$

$$\omega_1 = \frac{1}{2} \times \omega_2$$

$$2\omega_1 = \omega_2$$

$$\omega_2 = 2\omega_1$$

So, whilst velocity will be doubled.

3: Compare kinematics equation of linear motion and circular motion.

ANS:

RELATIONSHIP BETWEEN LINEAR AND ANGULARLY QUANTITATES

Consider an object revolving in circle, in order to find the relations between linear and angular quantities of the motion.

We know that:

Relation Between Angular Distance And Linear Distance:

$$\mathbf{S = r\,\theta}$$

$$\boldsymbol{\theta = \frac{S}{r}}$$

Relation Between Angular Velocity And Linear Velocity:

If ΔS is its distance for rotating through angle Δq then,

$$\boldsymbol{\Delta\theta = \frac{\Delta S}{r}}$$

Dividing both sides by Δt we get:

$$\boldsymbol{\frac{\Delta\theta}{\Delta t} = \left(\frac{\Delta S}{r}.\Delta t\right)}$$

$$\boldsymbol{\frac{r\Delta\theta}{\Delta t} = \frac{\Delta S}{\Delta t}}$$

If time interval Δt is very small Δt → 0, then the angle through which the particle moves is also very small and therefor the ratio $\Delta\theta/\Delta t$ gives the instantaneous angular speed ω_{ina}.

$$\boldsymbol{\lim_{\Delta t\to 0}\frac{\Delta S}{\Delta t} = r\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}}$$

Now by definition

$$\boldsymbol{v = \lim_{\Delta t\to 0}\frac{\Delta S}{\Delta t}}$$

$$\boldsymbol{\omega = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}}$$

Therefore equation (1) reduces to:

$$\mathbf{V = r\omega}$$

RELATION BETWEEN ANGULAR ACCELERATION AND LINEAR ACCELERATION

Suppose an object rotating about a fixed axis changes its angular velocity by Δω in time Δt sec, then the change in tangential velocity ΔV at the end of this interval will be

$$\mathbf{\Delta V = r\Delta\omega}$$

Change in velocity in unit time is given by:

$$\boldsymbol{\frac{\Delta V}{\Delta t} = r.\frac{\Delta\omega}{\Delta t}}$$

If Δt approaches to zero then ΔV/Δt will be instantaneous tangential acceleration and Δω/Δt will be instantaneous angular acceleration "α".

$$\boldsymbol{\alpha = r\alpha}$$

4: Can a small force ever exert a greater torque than a larger force? Give reason.

ANS: A small force can exert a greater torque than a larger force if the small force has a longer lever arm. Torque is defined as the product of force and lever arm, so a small force with a long lever arm can produce a torque that is greater than a large force with a short lever arm.

5: Give two real world applications of angular momentum.

ANSWER: Two real-world applications of angular momentum.

- **Figure skating**
  When a figure skater pulls their arms and legs in close to their body, they are decreasing their moment of inertia. This means that their angular velocity will increase. This is why figure skaters can spin faster when they are in a tight tuck position.

- **Gyroscopes**
  Gyroscopes are devices that use angular momentum to maintain their orientation. For example, a gyroscope can be used to keep a compass needle pointing north, even when the gyros cope is rotating.

6: Derive relationship between torque and angular acceleration.

ANS: This relationship between torque and angular acceleration indicates that the torque applied to an object is equal to the moment of inertia of the object multiplied by its angular acceleration. In a way, this equation is the rotational analogue to $F = m.a$ in linear motion, where F is force, m is mass, and a is linear acceleration.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\because \ \vec{F} = m\vec{a}$$

$$\vec{\tau} = \vec{r} \times m\vec{a}$$

$$\vec{\tau} = m(\vec{r} \times \vec{a})$$

7: List the moment of inertia dependent factors.

ANS: The moment of inertia of a rotating body depends on the following factors.

- **Mass**
- **Shape (Size)**
- **Distribution of mass**
- **Axis of rotation**

CHAPTER = 4 ROTATIONAL AND CIRCULAR MOTION

RESONING QUESTIONS

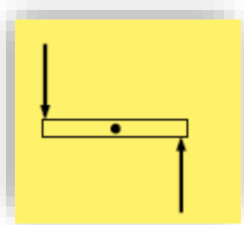
1. Can the sum of the torque on an object be zero while the net force on the object is nonzero? Explain

Ans. **If the sum of the forces on an object are not zero, then the CM of the object will accelerate in the direction of the net force. If the sum of the torques on the object are zero, then the object has no angular acceleration. Some examples are:**

i) A satellite in a circular orbit around the Earth.
ii) A block sliding down an inclined plane.
iii) An object that is in projectile motion but not rotating
iv) The startup motion of an elevator, changing from rest to having a non-zero velocity.

2. If the net force on a system is zero, is the net torque also zero? Explain

Ans. **Just because the net force on a system is zero, the net torque need not be zero. Consider a uniform object with two equal forces on it, and shown in the diagram. The net force on the object is zero (it would not start to translate under the action of these forces), but there is a net counterclockwise torque about the center of the rod (it would start to rotate under the action of these forces).**

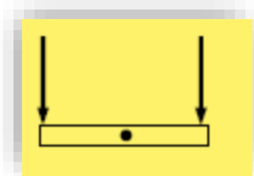


3. Why do divers and acrobats whish to make several somersaults pull their hands and feet close to their bodies?

Ans. **When divers and acrobats who wish to make several somersaults, they pull their hands and feet close to their bodies in order to rotate at the high rate. Due to the close distribution of mass the moment of inertia decreases. This cause an increase in the angular velocity enabling them to make several somersaults. The external force due to gravity acts through the center of gravity and hence no external torque about this point. Therefore the angular momentum about the center of gravity is constant.**

4. If the net torque on a system is zero, is the net force zero, Explain.

Ans. **Just because the net torque on a system is zero, the net force need not be zero. Consider an object with two equal forces on it, as shown in the second diagram. The net torque on the object is zero (it would not start to rotate under the action of these forces), but there is a net downward force on the rod (it would start to translate under the action of these forces).**



5. If a force F acts on an object such that its lever arm is zero, does it have any effect on the object's motion? Explain.

Ans. If the lever arm is zero, the force does not exert any torque and cannot produce an angular acceleration. There will be no change in the angular state of motion. However, the force will add to the net force on the body and so will change the linear acceleration of the body. The body's linear state of motion will change

6. In a translator motion, it is not necessary for a body to move in a straight line. Discuss the statement. (Karachi board 2012 Annual)

Ans. **The motion of a body said to be 'Translatory' if the axes of the frame of reference of a body remains always parallel to the corresponding axes of the observer's frame of reference. The motion of a body may not be necessarily along straight line in case of translational motion. i.e. The translational motion may or may not be linear motion. In the following figure shows translational motion of an object moving from A to B and C. Observe that throughout the motion every point of the object undergoes same displacement as every other point, it fulfill the condition of translational motion.**

7. Why does a slight push on an axle of wheel not cause any motion?
(Karachi board 2010 Annual)

Ans. According to the equation of maximum torque is given by

$$\tau = r F$$

Where

r = is the perpendicular distance from the point of rotation to the point of line of action of force (moment arm)

F = magnitude of the applied force

The above expression shows that the greater the force, larger will be the torque and larger the moment arm the greater will be the torque. It is therefore, easier to rotate a wheel by applying a given force when it is line of action is at greater distance from the center of rotation.

If the applied force on the axle of the wheel the wheel doesn't rotate because the moment arm is zero, torque also zero

$$\tau = r F$$

$$\tau = (o) F$$

$$\tau = o$$

8. Two boys ride on a marry-go-round, with boy 1 at a greater distance from the axis of rotation than the boy 2.

(i) Is the angular speed of boy 1 is (a) grater than (b) less than or (c) are same as the angular speed of boy 2

(ii) Is the tangential speed of boy 1 is (a) grater than (b) less than or (c) are same as the tangential of boy 2

Ans. (i) At any given time, the angle θ for boy 1 is the same as the angle of boy 2. As a result, they have the same angular speed. Every point on the marry-go-round has the same angular speed.

(ii) The tangential speeds are different, however, boy 1 has the greater tangential speed since he travels around a larger circle in the same time

Boy 2 travels around a small circle. This is in agreement with $v = r\omega$, since boy 1 has the larger radius.

9. Why is it easy to turn along a curved path of a large radius compared to a curved path of a short radius?

Ans. When a body moves along a curved path, a centripetal force acts on it and is given by

$$F_c = \frac{mv^2}{r}$$

As centripetal force is inversely related to the radius, i.e. if radius is large then less force is required and if radius is short then large force is required. So it is easy to turn along a curved path of large radius as compared to curved path of short radius.

10. In what direction does mud fly off the tyre of a moving bicycle?

Ans. As mud on a tyre moves, it requires centripetal force. This force is supplied by the force of adhesion between mud and tyre. When the speed of the wheel increases, the centripetal force also increases. At a certain speed, the force of adhesion becomes insufficient to supply the necessary centripetal force. Thus mud flies off tangentially along a straight path.

11 Define angular velocity. Give the units. Establish the relation V= r ω (Karachi board 2012 Annual)

ANGULAR VELOCITY:

The rate of change of angular displacement is called angular velocity or angular frequency.It is a vector quantity represented by

UNITS:

Its units are

(i) rad / sec (ii) deg / sec (iii) rev/sec

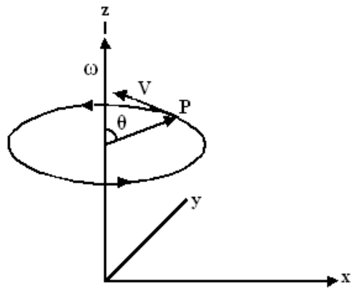
RELATION BETWEEN LINEAR VELOCITY V ND ANGULAR VELOCITY ω:

Consider a particle 'P' in a body rotating in a circle of radius 'r' as shown. Suppose the particle 'P' moves through a distance along the arc when the body rotates through a distance along the arc when the body rotates through an angle, such that

$$\Delta S = r\Delta\theta$$

Dividing both side by the time interval in which the rotation occurred, we get

$$\frac{\Delta S}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$



Applying $\underset{\Delta t \to 0}{\text{Limit}}$ on both sides, above equation becomes

$$\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta S}{\Delta t} = \underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\theta}{\Delta t} \quad \text{----------- (i)}$$

Where $\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta S}{\Delta t}$ = instantaneous linear velocity

$$\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta S}{\Delta t} = V$$

$\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\theta}{\Delta t}$ = instantaneous angular velocity

$$\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\theta}{\Delta t} = \omega$$

substituting these values in equation (1), we get

$$V = r\,\omega$$